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An inhomogeneous magnetohydrostatic universe

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Abstract. An inhomogeneous magnetohydrostatic cosmological model has been derived which is of Petrov type 1D.

1. Introduction

Static homogeneous cosmological models were derived by Einstein and de Sitter in which the material distribution is that of a perfect fluid. A magnetic universe with matter has been studied by Khalatnikov (1967). In recent years a cosmological model describing a static magnetic universe has been derived by Patel and Vaidya (1971). A class of inhomogeneous cosmological models has been derived by Szekeres (1975). A variety of exact cosmological solutions of the Einstein field equations with perfect fluid source have been studied by Wainwright *et al* (1979). Non-static cosmological models of different Bianchi types have been obtained by Jacobs (1969) and Jacobs and Hughston (1970). A detailed study of homogeneous cosmological models containing a magnetic field has been presented by Thorne (1967). In our previous paper we have derived an anisotropic magnetohydrodynamic cosmological model (Roy and Prakash 1978). In this paper we have constructed a magnetohydrostatic model in which the material distribution is given by a perfect fluid together with an incident magnetic field, the free gravitational field of which is of Petrov type 1 degenerate (1D). It is found that the model is an inhomogeneous one and the inhomogeneity is due to the introduction of the magnetic field.

2. Derivation of the line element

We consider the cylindrically symmetric metric in the form

$$ds^2 = A^2(dx^2 - dt^2) + B^2\, dy^2 + C^2\, dz^2 \tag{2.1}$$

where the metric potentials A, B and C are functions of x only. The distribution consists of an electrically neutral perfect fluid with an infinite electrical conductivity and a magnetic field. The energy momentum tensor of the composite field is assumed to be the sum of the corresponding energy momentum tensors. Thus

$$T_i^j = (\epsilon + p) V_i V^j + p\delta_i^j + E_i^j \tag{2.2}$$

where

$$E_i^j = \mu[h_l h^l (V_i V^j + \tfrac{1}{2}\delta_i^j) - h_i h^j], \tag{2.3}$$

μ being the magnetic permeability, ϵ the density, p the pressure and $h_i = \mu^{-1} V^{j*} F_{ji}$ (Lichnerowicz 1967, page 93). F_{ij} is the electromagnetic field tensor and V^i is the flow vector satisfying

$$g_{ij} V^i V^j = -1. \tag{2.4}$$

The coordinates are assumed to be comoving so that

$$V^1 = V^2 = V^3 = 0 \qquad \text{and} \qquad V^4 = 1/A.$$

We assume the incident magnetic field to be in the direction of the x axis so that F_{23} is the only non-vanishing component of the tensor F_{ij}. The first set of Maxwell equations leads to F_{23} being a constant, say H.

The field equations

$$R_i^j - \tfrac{1}{2} R \delta_i^j + \Lambda \delta_i^j = -8\pi T_i^j \tag{2.5}$$

for the line element (2.1) are as follows:

$$\frac{1}{A^2}\left(\frac{A_1 B_1}{AB} + \frac{B_1 C_1}{BC} + \frac{A_1 C_1}{AC}\right) - \Lambda = 8\pi\left(p - \frac{H^2}{2\mu B^2 C^2}\right) \tag{2.6}$$

$$\frac{1}{A^2}\left(\frac{C_{11}}{C} + \frac{A_{11}}{A} - \frac{A_1^2}{A^2}\right) - \Lambda = 8\pi\left(p + \frac{H^2}{2\mu B^2 C^2}\right) \tag{2.7}$$

$$\frac{1}{A^2}\left(\frac{B_{11}}{B} + \frac{A_{11}}{A} - \frac{A_1^2}{A^2}\right) - \Lambda = 8\pi\left(p + \frac{H^2}{2\mu B^2 C^2}\right) \tag{2.8}$$

$$\frac{1}{A^2}\left(-\frac{B_{11}}{B} - \frac{C_{11}}{C} - \frac{B_1 C_1}{BC} + \frac{A_1 B_1}{AB} + \frac{A_1 C_1}{AC}\right) + \Lambda = 8\pi\left(\epsilon + \frac{H^2}{2\mu B^2 C^2}\right). \tag{2.9}$$

The non-vanishing components of the Weyl conformal curvature tensor $C_{hi}{}^{jk}$ for the metric (2.1) are

$$\begin{aligned}
C_{14}{}^{14} = C_{23}{}^{23} &= \frac{1}{6A^2}\left(-\frac{B_{11}}{B} - \frac{C_{11}}{C} + 2\frac{A_{11}}{A} - 2\frac{A_1^2}{A^2} + 2\frac{B_1 C_1}{BC}\right) \\
C_{12}{}^{12} = C_{34}{}^{34} &= \frac{1}{6A^2}\left(-\frac{A_{11}}{A} + 2\frac{B_{11}}{B} - \frac{C_{11}}{C} + 3\frac{A_1 C_1}{AC} + \frac{A_1^2}{A^2} - 3\frac{A_1 B_1}{AB} - \frac{B_1 C_1}{BC}\right) \\
C_{13}{}^{13} = C_{24}{}^{24} &= \frac{1}{6A^2}\left(-\frac{A_{11}}{A} + 2\frac{C_{11}}{C} - \frac{B_{11}}{B} + 3\frac{A_1 B_1}{AB} + \frac{A_1^2}{A^2} - 3\frac{A_1 C_1}{AC} - \frac{B_1 C_1}{BC}\right).
\end{aligned} \tag{2.10}$$

The suffix 1 after the symbols A, B and C indicates ordinary differentiation with respect to x. Equations (2.6)–(2.9) are four equations in five unknowns (A, B, C, ϵ and p). For the complete determination of these unknowns one more condition has to be imposed on them. Here we assume that the space–time is of degenerate Petrov type 1, the degeneracy being in the y and z directions. This requires that $C_{12}{}^{12} = C_{13}{}^{13}$. Thus we have

$$\frac{B_{11}}{B} - \frac{C_{11}}{C} - 2\frac{A_1}{A}\left(\frac{B_1}{B} - \frac{C_1}{C}\right) = 0. \tag{2.11}$$

Equation (2.11) is identically satisfied if $B = C$. However, we shall assume the metric potentials to be unequal. From equations (2.6) and (2.7) we obtain

$$\frac{B_1C_1}{BC}+\frac{A_1B_1}{AB}+\frac{A_1C_1}{AC}-\frac{C_{11}}{C}-\frac{A_{11}}{A}+\frac{A_1^2}{A^2}=-\frac{8\pi H^2A^2}{\mu B^2C^2}. \tag{2.12}$$

Also, from equations (2.7) and (2.8), we get

$$\frac{B_{11}}{B}-\frac{C_{11}}{C}=0. \tag{2.13}$$

Equations (2.11) and (2.13) lead to

$$\frac{A_1}{A}\left(\frac{C_1}{C}-\frac{B_1}{B}\right)=0. \tag{2.14}$$

Since $B \neq C$, on integration, equation (2.14) gives

$$A=\text{constant}=M. \tag{2.15}$$

From equations (2.12) and (2.15) we get

$$\frac{B_1C_1}{BC}-\frac{C_{11}}{C}=-\frac{8\pi H^2M^2}{\mu B^2C^2}. \tag{2.16}$$

On integration, equation (2.13) gives

$$B_1C-BC_1=K \tag{2.17}$$

where K is a constant of integration.

Let $BC=\alpha$ and $B/C=\beta$, so that $B^2=\alpha\beta$ and $C^2=\alpha/\beta$; we have from equation (2.17)

$$[\beta]_1=K\beta/\alpha. \tag{2.18}$$

From equations (2.16) and (2.18) we get

$$\alpha\alpha_{11}-\alpha_1^2+K^2-\frac{16\pi H^2M^2}{\mu}=0. \tag{2.19}$$

On integration, equation (2.18) gives

$$\alpha=n\sqrt{l}\sin\left(\frac{x+b}{\sqrt{l}}\right) \tag{2.20}$$

where b and l are constants of integration and

$$n=\left(K^2-\frac{16\pi H^2M}{\mu}\right)^{1/2}.$$

From equations (2.18) and (2.20) we get

$$\beta=m\left(\tan\left(\frac{x+b}{2\sqrt{l}}\right)\right)^{K/n} \tag{2.21}$$

where m is a constant of integration. Hence

$$B^2=mn\sqrt{l}\sin\left(\frac{x+b}{\sqrt{l}}\right)\left(\tan\left(\frac{x+b}{2\sqrt{l}}\right)\right)^{K/n} \tag{2.22}$$

and

$$C^2=\frac{n\sqrt{l}}{m}\sin\left(\frac{x+b}{\sqrt{l}}\right)\left(\tan\left(\frac{x+b}{2\sqrt{l}}\right)\right)^{-K/n}. \qquad (2.23)$$

Consequently the line element (2.1) takes the form

$$\mathrm{d}s^2=M^2(\mathrm{d}x^2-\mathrm{d}t^2)+mn\sqrt{l}\sin\left(\frac{x+b}{\sqrt{l}}\right)\left(\tan\left(\frac{x+b}{2\sqrt{l}}\right)\right)^{K/n}\mathrm{d}y^2$$
$$+\frac{n\sqrt{l}}{m}\sin\left(\frac{x+b}{\sqrt{l}}\right)\left(\tan\left(\frac{x+b}{2\sqrt{l}}\right)\right)^{-K/n}\mathrm{d}z^2. \qquad (2.24)$$

Using the transformations

$$\begin{aligned}&\sin\left(\frac{x+b}{2\sqrt{l}}\right)=\frac{\bar\rho}{\rho}\\ &\frac{\sqrt{m}\sqrt{n}(\bar\rho)^{K/2n}}{\sqrt{M}}y=\phi\\ &\frac{\sqrt{n}}{\sqrt{m}\sqrt{M}(\bar\rho)^{K/2n}}z=Z\\ &Mt=T\end{aligned} \qquad (2.25)$$

the metric (2.24) can be put into the form

$$\mathrm{d}s^2=\frac{\mathrm{d}\rho^2}{\rho^2(\rho^2-\bar\rho^2)}-\mathrm{d}T^2+\frac{1}{\rho^2}(\rho^2-\bar\rho^2)^{(n-K)/2n}\mathrm{d}\phi^2+\frac{1}{\rho^2}(\rho^2-\bar\rho^2)^{(n+K)/2n}\mathrm{d}Z^2 \qquad (2.26)$$

where $\bar\rho^2=1/4M^2l$.

3. Some physical and geometrical features

The distribution in the model is given by

$$8\pi p=-\bar\rho^2-\Lambda \qquad (3.1)$$

$$8\pi\epsilon=-\bar\rho^2\left[2\frac{K^2}{n^2}-5+2\left(\frac{K^2}{n^2}-1\right)\frac{(\rho^2-2\bar\rho^2)^2}{4\bar\rho^2(\rho^2-\bar\rho^2)}\right]+\Lambda. \qquad (3.2)$$

The model has to satisfy the reality conditions (Ellis 1971, page 117)

(*a*) $(\epsilon+p)>0$ and

(*b*) $(\epsilon+3p)>0$

which require

(i) $n^2<K<2n^2$,

(ii) $\Lambda\leq-\bar\rho^2/n^2$ and

(iii) $(\sqrt{2}/\alpha)\bar\rho[1-(1-\alpha^2)^{1/2}]^{1/2}<\rho<(\sqrt{2}/\alpha)\bar\rho[1+(1-\alpha^2)^{1/2}]^{1/2}$

where α^2 stands for the greater of $(K^2-n^2)/n^2$ and $-(K^2-n^2)\bar\rho^2/\Lambda n^2$. The non-vanishing components of the conformal curvature tensor C_{hijk} are

$$C_{12}^{\;12}=C_{13}^{\;13}=-\tfrac{1}{2}C_{23}^{\;23}=\frac{1}{3}\left[\left(\frac{K^2}{n^2}-1\right)\frac{\rho^4}{4(\rho^2-\bar\rho^2)}\right]. \qquad (3.4)$$

Clearly there is a real singularity at $\rho = \bar{\rho}$ in the model (2.26). The space characterised by (2.26) is therefore valid for $\rho > \bar{\rho}$. The region $\rho \leqslant \bar{\rho}$ is occupied by the source of the magnetic field. Obviously the field is divergent at $\rho = \infty$, this being due to the infinite nature of the source. From equations (3.1), (3.2) and (2.9) it is clear that the hydrostatic pressure and the total density are constant, although the matter density and the magnetic energy density separately diverge as $\rho \to \infty$. It is to be noted that the magnetic field introduces an inhomogeneity only in the density, the pressure being uniform. The components of the electromagnetic field tensor F_{ij} are constant in the present coordinate system but the tensor is not covariant constant. We therefore do not call it a constant magnetic field. The metric (2.26) admits a group of G_3 of isometries that is an Abelian group transitive on three-dimensional time-like surfaces. The kinematical parameters for the flow vector V^i are all zero.

In the absence of the magnetic field the model of the space–time is given by

$$ds^2 = \frac{d\rho^2}{\rho^2(\rho^2 - \bar{\rho}^2)} - dT^2 + \frac{1}{\rho^2} d\phi^2 + \frac{\rho^2 - \bar{\rho}^2}{\rho^2} dZ^2, \tag{3.5}$$

which represents the Einstein universe. It is worth noticing that for the solution (2.26) the equation of state of the fluid, as well as the electromagnetic field, is inhomogeneous; but the fluid flow is kinematically the same as that in the Einstein static universe. So the model can be described as an Einstein static universe made inhomogeneous by addition of a magnetic field, and with an inhomogeneous physical condition.

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